NON-SMOOTH PROBLEMS OF MECHANICS OF MACHINES FOR CONSTRUCTION

Wiesław Grzesikiewicz, Jerzy Piotrowski

Warsaw University of Technology, Institute of Vehicles, ul. Narbutta 84, 02-524 Warszawa, Poland. E-mail: jpt@simr.pw.edu.pl

Abstract: This paper presents a mathematical formulation of non-smooth problem of mechanics. Some non-smooth characteristics of chosen elements are presented on the diagrams. These elements serve the purpose of representing unilateral limiters of displacements and velocities, dry friction and pseudo-elasticity. The main feature of the non-smooth problem is that for its formulation the non-differentiable functionals have to be used. In the result, in the mathematical description of motion of the system there appear multi-valued functions. Some examples of modelling of machines with non-smooth elements have been presented.

Keywords: non-smooth mechanics, dry friction, limiters, machines for construction

1. INTRODUCTION

The problems of mechanics, which for their formulation require non-differentiable functionals, are referred to as non-smooth ones. For example in static problems of mechanical systems, the use is made of the potential energy. If there are bump-stops in the system (unilateral constraints) or the pseudo-elastic elements, then the functional of energy is non-differentiable. The position of equilibrium of such a system is determined by a solution of a non-smooth static problem.

The motion of the mechanical system with unilateral constraints or/and dry friction is described by the non-smooth problem of dynamics. In such a case the functional of potential energy and functional of dissipation are non-differentiable.

To model machines for construction, the mechanical systems with described above properties are used. Therefore, the description of motion or loading of the machine with bump-stops, clearances or dry friction should be formulated as a non-smooth problem of mechanics [3]. A specific feature of the description of a non-smooth problem is that such functions are involved, whose values are not uniquely determined but belong to some range of values. Additionally, a problem of the unique choice of the value from a defined range arises. To describe mathematically such problems the notions of convex analysis are employed [1]. We present some of these notions within a treatment of a non-smooth static problem with unilateral constraints and pseudo-elastic elements.

2. NON-SMOOTH STATIC PROBLEMS

Let us consider a mechanical system, whose configuration is described by a vector of generalised co-ordinates $X \in R^N$. We assume that the potential energy of the system without constraints and pseudo-elastic elements is described by the expression

$$E_o(X) := \frac{1}{2} X^T K X - F^T X \tag{1}$$

where

$K \in R^{N \times N}$ - a symmetric, strictly positive matrix of rigidity;

$F \in R^N$ **-** vector of generalised forces**.**

The static problem reduces to determination of the position of equilibrium. A classical problem, where the potential energy describes equation (1) is formulated as follows:

find such a vector $X \in R^N$*, that the functional* E_o *takes the minimal value*

$$X = arg \min_{\xi \in R^N} E_o(\xi) \ . \tag{2}$$

Making use of the Lagrange condition, the problem (2) can be presented in an equivalent algebraic form:

find a vector $X \in R^N$ *satisfying the equation of equilibrium of forces*

$$KX = F. \tag{3}$$

Now, we are going to describe mathematically properties, which make that the static problem is non-smooth. The limiters (constraints) define permissible positions of the system. These positions can be described with a closed convex set $\Omega \subset \mathrm{R}^N$, and the limitation with the condition

$$X \in \Omega . \tag{4}$$

Reaction forces $r \in \mathrm{R}^N$ execute the limitation of the positions. If we assume that execution is ideal, then the potential of reaction forces is determined by non-differentiable, indicatory functional of the set Ω, i.e.:

$$\psi_\Omega(X) := \begin{cases} 0 & ,if \quad X \in \Omega; \\ +\infty & ,if \quad X \notin \Omega, \end{cases} \tag{5}$$

where ψ_Ω - is the potential of reaction forces.

The pseudo-elasticity is described by a set of forces $\Theta \subset \mathrm{R}^N$ and the principle of choosing the force from the set. If we assume that the set is convex and closed and the force $S \in \Theta$ is chosen in an ideal manner, then the potential of the pseudo-elasticity is defined by a non-differentiable functional supporting the set Θ

$$\sigma_\Theta(X) := \sup_{S \in \Theta} S^T X \ , \tag{6}$$

where $\Theta \subset \mathrm{R}^N$ -is the set of pseudo-elastic forces.

The potential of a mechanical system with constraints and pseudo-elasticity is described by a functional containing components listed in expressions (1) ,(5) and (6), i.e.:

$$\mathrm{E}(X) = \frac{1}{2} X^T K X - F^T X + \psi_\Omega(X) + \sigma_\Theta(X) \ . \tag{7}$$

The formulation of a non-smooth static problem, whose energy describes the non-differentiable functional F, has an analogous form to that given by (2):

find a vector $X \in \mathrm{R}^N$, *such that the functional* E *takes a minimal value*

$$X = arg \min_{\xi \in \mathrm{R}^N} \mathrm{E}(\xi) \ . \tag{8}$$

Due to assumptions referring to the matrix K and sets Ω i Θ, the above problem has a unique solution. Making use of the Kuhn-Tucker conditions [4], the problem (8) can be described in the equivalent algebraic form, analogous to expression (3)

$$\begin{cases} KX + S + r = F; \ (9a) \\ r \in \partial \psi_\Omega(X), X \in \Omega; \ (9b) \\ S \in \partial \ \sigma_\Theta(X), S \in \Theta; \ (9c) \end{cases}$$

where ∂ is a symbol of sub-derivative of a convex functional, that is, if $\Phi : \mathrm{R}^N \to \mathrm{R}^1$ is a convex functional, then

$$\partial \ \Phi(z) := \left\{ f \in R^N : \Phi(\zeta) - \Phi(z) \geq f^T(\zeta - z), \forall \zeta \in R^N \right\} \ . \tag{10}$$

The equation (9a) presents a condition of equilibrium of forces acting on the system and the set (9b) and (9c) describes relations between reaction force or pseudo-elasticity and the position of the system. These relations are referred to as the energetic characteristics of constraints or pseudo-elasticity. The form of these relations decides upon the specific feature of non-smooth formulation of the static problem, where the forces r and S are described by multi-valued functions of positions. This specific form of functions requires non-classical methods of solution.

3. THE ELEMENTS OF MECHANICAL SYSTEMS WITH NON-SMOOTH CHARACTERISTICS

In the previous chapter a general form of description of non-smooth characteristics of the mechanical system has been presented. We refer to formulae (5) and (6) presenting the non-differentiable functionals describing the potential of generalised, ideal reaction force and pseudo-elasticity. The equivalent form of description of these features is included in formulae (9b) and (9c), where the relations are given between said forces and the position of the system. These relations are called the generalised characteristics of unilateral constraints and pseudo-elasticity. The generalised form of the description, i.e. presented by the generalised co-ordinates, is formulated on the basis of description of the elements included in the system. Below we present the basic elements with non-smooth characteristics.

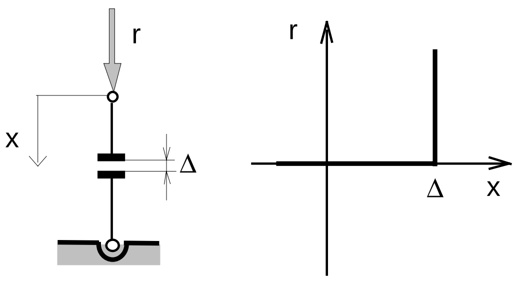

Figure 1: A unilateral limiter

In Figure 1 a symbol of a limiter is shown, which allows displacements satisfying condition

$$x \in \Omega\,, \quad \Omega := \left\{ x \in R^1 \;:\; x \le \Delta \right\}.$$

On the right side of the drawing the plot of the characteristic is shown, which is the relation between the reaction force r and the displacement x. When $x = \Delta x$, the value of the reaction force r is not uniquely defined as it belongs to a set of non-negative numbers. Therefore, the mathematical description of this characteristic can be written in the form

$$r = 0\,, \quad \text{gdy} \quad x < \Delta\,;$$

$$r \ge 0\,, \quad \text{gdy} \quad x = \Delta\,.$$

A symbol of a double limiter and the plot of its characteristic are shown in Figure 2. In this case the permissible displacements satisfy the condition

$$x \in \Omega, \Omega := \left\{ x \in R^1 : -\Delta \le x \le \Delta \right\}$$

A symbol of and the characteristic of the pseudo-elastic element are presented in Figure 3.

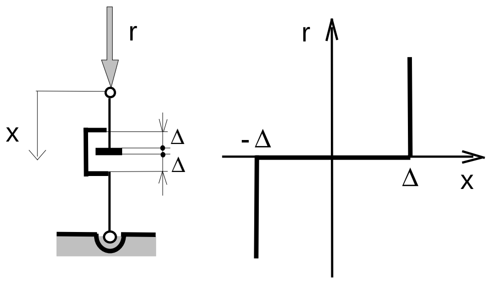

Figure 2: A double limiter with clearances

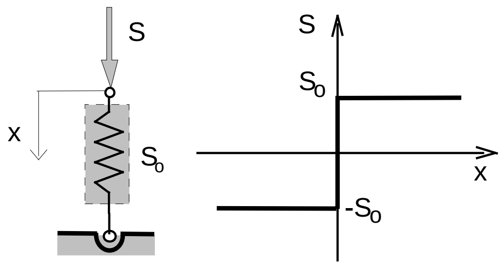

Figure3: Pseudo-elastic element

A specific feature of this element is, that the force S should satisfy the following condition

$$S \in \Theta\,, \quad \Theta := \left\{ S \in R^1 \;:\; -S_o \le S \le S_o \right\}.$$

This element serves to reproduce in modelling the devices with pre-loaded spring. A schematic drawing of such a device and its model are shown in Figure 4.

All described elements characterise the energetic features of the mechanical system related to accumulation of the potential energy.

The mechanical systems frequently contain elements with non-smooth characteristics, which are responsible for dissipation of energy. Dry friction and/or the velocity limiters are reproduced in modelling with such elements.

The ability of the mechanical system to dissipate energy determines the functional of dissipation (the function of Rayleigh). If a mechanical system has properties represented by the above elements, then the functional is not differentiable and its form is analogous to that given by (7).

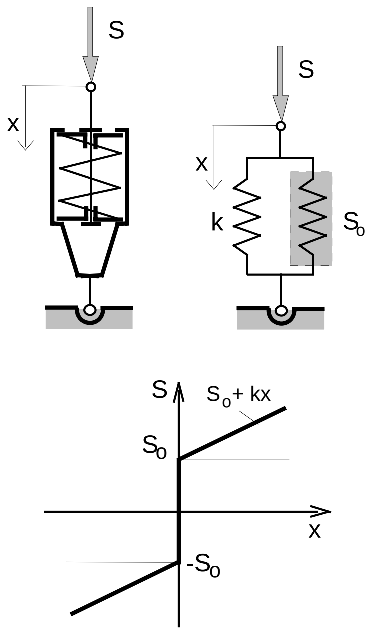

Figure 4: Pseudo-elastic element combined with spring

$$D(V) := D_o(V) + \psi_{\Omega_d}(V) + \sigma_{\Theta_d}(V)\,, \quad (11)$$

where

$V \in R^N$ - generalised velocities vector;

$D : R^N \to R^1$ - functional of dissipation;

$D_o : R^N \to R^1$ - functional defining differentiable part of D;

$\Omega_d \subset R^N$ - set defining permissible velocities;

$\Theta_d \subset R^N$ - set defining permissible values of dry friction forces.

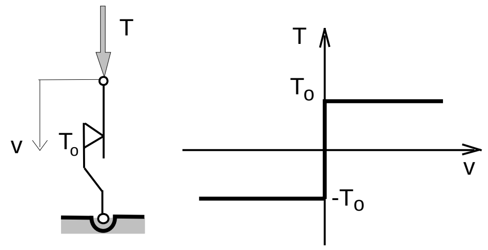

Figure 5: Dry friction slider

The dissipative characteristic of the damping element defines the relation between the force and velocity. Figure 5 shows the graphical symbol of friction element and the diagram of its characteristic.

The force arising in this element satisfies the condition

$$T \in \Theta_d\ , \quad \Theta_d := \left\{ T \in R^1 :\ -T_o \le T \le T_o \right\}.$$

The graphical symbol of the velocity limiter and its characteristic are presented in Figure 6. The limiter allows for velocities satisfying the following condition

$$v \in \Omega_d\ , \quad \Omega_d := \left\{ v \in R^1 \quad : \quad v \le 0 \right\}.$$

Described above basic elements with non-smooth characteristics are used to represent the

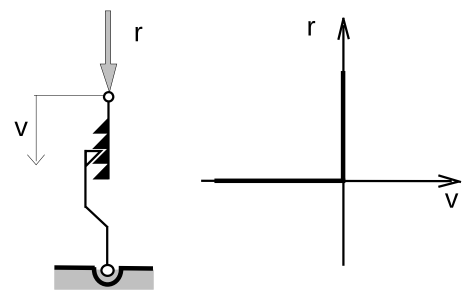

Figure 6: Velocity limiter

characteristics of various units and elements of machines [2].

4. EXAMPLES OF MACHINE DRIVING SYSTEM MODELLING

As the first example the simplest driving system is considered, which consists of two flywheels coupled by the friction clutch. The schematic drawing of the system and the clutch characteristic are shown in Figure 7. With controlled normal load of the clutch disks, the moment $M_o(t)$ is not constant.

$$\begin{cases} J_1 \dot{\omega}_1 = M_n - M_T\,; \\ J_2 \dot{\omega}_2 = -M_{ob} + M_T\,; \end{cases} \qquad (12a)$$

$$\begin{cases} M_T := \tau \ \ M_o(t), \omega_s := \omega_1 - \omega_2, \tau \in [-1,+1]; (12b) \\ \tau = \Pi(\tau + \rho\omega_s); \quad (12c) \end{cases}$$

where:

J_1, J_2 - moments of inertia of flywheels;
ω_1, ω_2 - angular velocities;
M_n, M_{ob} - driving and opposing moments respectively;
M_T - moment transmitted by clutch;
ω_s - angular slip velocity of the clutch;
τ - friction moment multiplier;
ρ - arbitrary positive number;
$\Pi(.)$ -a function defined by

$$\Pi(z) := \begin{cases} z & ,if \quad |z| \le 1, \\ sign \quad z & ,if \quad |z| > 1. \end{cases} \qquad (13)$$

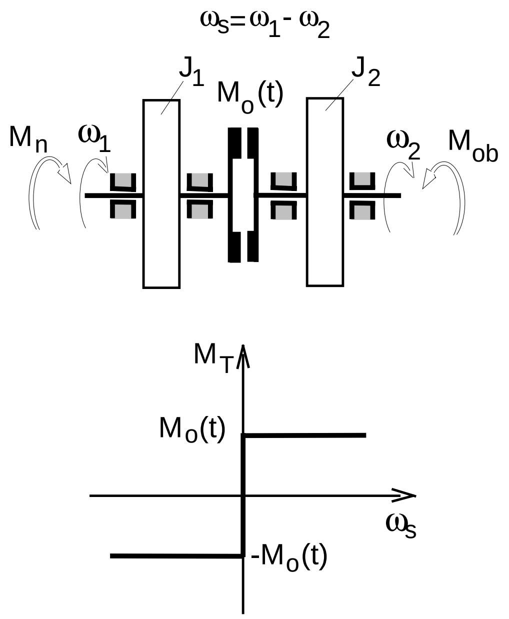

Figure 7: A transmission system with friction clutch

The equations (12a) describe motion of wheels. The relations (12b), (12c) describe the friction clutch with the characteristic shown in Figure 7. Analysing the plot of the characteristic or the equivalent description (12c), it is easy to notice that if $\omega_s = 0$, then the moment transmitted by the clutch is defined by the inequality $-M_o(t) \le M_T \le M_o(t)$. To define the value of the moment M_T in this situation, it is necessary to complete the set of equation (12) with the differential succession of the clutch characteristic, which has a form [3]

$$\tau = \Pi(\tau + \rho\dot{\omega}_s), gdy\,\omega_s = 0\,. \qquad (14)$$

The equations (12) and (14) make it possible to uniquely determine the moment transmitted by the clutch for any velocity, i.e.: if $\omega_S \neq 0$, then $M_T = \operatorname{sign}\omega_S \cdot M_0(t)$, else $(\omega_S = 0)$, M_T is determined by (12a), (12b), and (14).

This simple example illustrates the principle of description of the system containing elements with non-smooth characteristic. A peculiarity of the problem is that in some states of the system (here $\omega_S = 0$) the description becomes non-unique and it is necessary to formulate the additional conditions, which make the description unique.

Using the same principle it is possible to formulate the description of motion of the driving system shown in Figure 8. In this example the flywheels are coupled by the ratchet-wheel with the characteristic as in the right side of Figure 8. The ratchet-wheel corresponds to the velocity limiter of Figure 6.

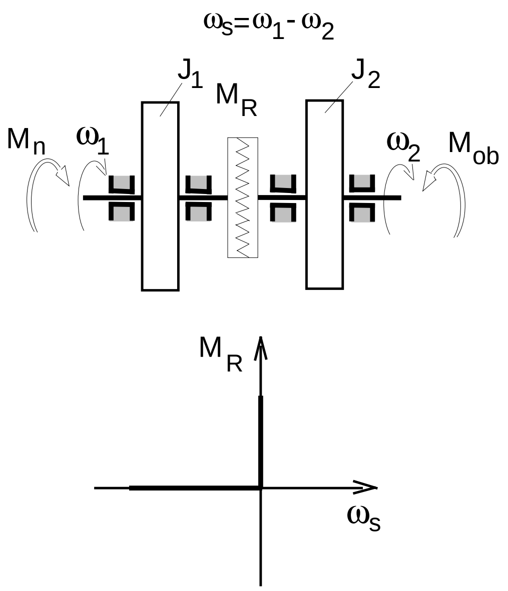

Figure 8: Transmission system with velocity limiter (ratchet wheel)

The equations of motion have a form

$$\begin{cases} J_1\dot{\omega}_1 = M_n - M_R\,, \\ J_2\dot{\omega}_2 = -M_{ob} + M_R\,, \end{cases} \qquad (15a)$$

$$M_R = \left[M_R - \rho\omega_S\right]^+\,, \quad \omega_S := \omega_1 - \omega_2\,, \qquad (15b)$$

where

M_R - moment transmitted by ratchet wheel,

$[\cdot]^+$ - a function, such that

$$[z]^+ := \begin{cases} 0 & ,gdy \quad z \leq 0, \\ z & ,gdy \quad z > 0. \end{cases} \qquad (16)$$

The equation (15b) describes the characteristic of Figure 8. The meaning of variables and parameters is given after equations (12).

The equations (15) and (16) make it possible to determine motion of flywheels and the moment transmitted by the ratchet-wheel.

5. FREE VIBRATIONS OF A SIMPLE SUSPENSION SYSTEM WITH DRY FRICTION DAMPER

This simple example is concerned with a system where the dry friction element of Figure 5 (a friction slider) is combined with springs and serves as a damper in a suspension system shown in Figure 9.

The equations of free vibrations of the system are written for the system of Figure 9b, where the spring k_1 and friction slider are replaced by the force of friction T.

$$M\ddot{Y} + kY = T\,,$$

$$\dot{T} = \begin{cases} k_1(-\dot{Y}) & if \quad |T| < T_o \\ -\left[-k_1(-\dot{Y})\right]^+ & if \quad T = +T_o \\ \left[k_1(-\dot{Y})\right]^+ & if \quad T = -T_o \end{cases}.$$

This description involves differential succession of the friction force T. The function $[\,.\,]^+$ has the form (16).

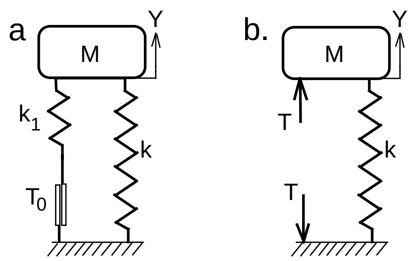

Figure 9: a) Simple mechanical model
b) Replacement system

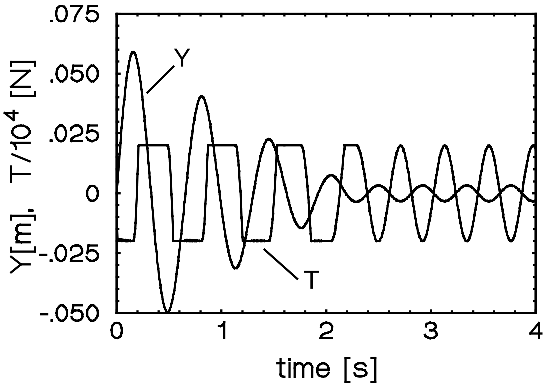

Figure 10: Time histories of free vibrations

A sample numerical solution has been calculated for mass M=450kg. Other parameters are: $T_o = 200\,N$, $k = 40000\,N/m$, $k_1 = 60000$, N/m. The initial conditions are:

for t = 0, Y= 0, dY/dt = 0.6 m/s, T= 0.

The equations of motion have been numerically integrated using a simple time-stepping routine with a constant time step of integration. The time histories of free vibrations are plotted in Figure 10. The plots present the displacement Y and the friction force T.

Initially, due to damping by dry friction, vibrations decay with a linear envelope. During this period there is reciprocating sliding in the friction slider causing dissipation. When displacements become small enough, sliding and dissipation cease but the force in the stuck slider oscillates as it is induced by deflections of the spring in series with the slider. The amplitude of the force T is marginally smaller than the break force T_o.

6. FINAL REMARK

Presented descriptions of non-smooth problems of mechanics have been used to work out mathematical models of machines, where dry friction or the limiters influence machine motion and loading. The significance of formulated problems for analysis of machines is especially pronounced for extreme operating conditions.

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