Mechanics of Tensegrity Prisms

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ABSTRACT

Tensegrity prisms are three-dimensional self-stressing cable systems with a relatively small number of disjoint compression members, invented by Buckminster Fuller. They form novel structural geometries and they constitute a class of mechanisms that have not been previously studied for possible application as variable geometry truss (VGT) manipulators. They have a number of seemingly advantageous properties -- they are self-erecting, in that tensioning the final cable transforms them from a compact group of members into a large three-dimensional volume, and they are predominately tension systems, in that they can function as a VGT manipulator while actuating members only in tension. These properties have not been explored but could be broadly useful, for applications ranging from temporary terrestrial construction to large on-orbit space structures. However, they have a number of properties that make them seemingly inappropriate for use -- they are not conventionally rigid, they exist only under specific conditions of geometry with a corresponding prestress state, and the governing equations that do exist include singular (non-invertible) matrices. In our opinion the advantages and application potential justify the study and discussion of tensegrity behavior. The mathematics of tensegrity geometry, statics, and kinematics have not been fully formulated, and such mathematical results must be developed and assembled before applications can be undertaken. This paper describes the physical behavior of a basic family of tensegrity prisms, presents the most useful available mathematical results, and outlines a preliminary simulation study of such a prism used as a VGT manipulator.

INTRODUCTION

As demonstrated by Buckminster Fuller and as popularized in artists' installations and in educational toys, tensegrity structures present a novel, distinctive geometry. The word tensegrity is derived from the phrase *tensile integrity*, referring to the fact that the

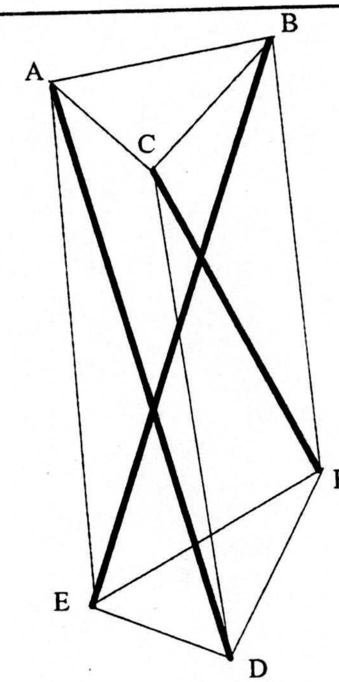

Figure 1. T-3 Tensegrity Prism

compression members in the typical structure are generally disjoint (not connected to one another) and that the only "continuous" paths connecting all nodes are formed by tension members; it is also noted that a prestress state can be sustained in such structures. These original defining principles are provocative, and by featuring the use of tension members they suggest possible material efficiencies. Moreover, any direct experimentation at constructing such structures reveals additional intriguing properties such as a self-erecting behavior and a range of adaptive geometry. Fundamentally, these are form-finding tension structures, and as such they are simple to construct but difficult to describe mathematically. Specifically, the precise structural, mechanical, and mathematical properties of such structures were not identified or recognized at the outset, and all of those questions remain largely open at this time. This paper reviews the available understandings and discusses the possible use of tensegrity structures as variable geometry trusses.

There are a large number of possible topologies (node connectivities) with which tensegrity structures can be built, and their devotees most frequently show them in complicated geometries such as a truncated icosahedron with 36 nodes, 18 (compression) bars, and 54 (tension) cables. We restrict this paper to a particular minimal form, in which three cables and one bar meet at each node. The tensegrity structure with the connectivity depicted in Figure 1 has nine cables and three bars. We refer to it as a T-3 prism, and our specific results will be offered for that topology; we will also discuss a related T-4 prism with 12 cables and 4 bars. Intriguing characteristics of such a prism include the fact that it transforms from a compact bundle of bars into a full three-dimensional framework as the last cable is pulled in tension, that it is a form-finding structure, that it becomes a prestressed system, that our conventional simple use of Maxwell's rule does not properly describe the structural type, that our conventional definitions of statical stability and statical determinacy do not directly apply, that our conventional model of structural stiffness does not apply, and so on. It has long been understood that by suitably changing the cable lengths the prism can be made to change its nodal geometry, and it is for this behavior that the prisms are likened here to variable geometry trusses.

We note at the outset that one cannot prescribe arbitrarily the nodal positions (the "placement") to define a tensegrity state. Equivalently, one cannot prescribe a set of member lengths that will construct a tensegrity structure. The topology depicted in Figure 1 is valid, but that tensegrity structure cannot generally be achieved at any geometry. However, from conditions of symmetry and statics, one starting state can be defined mathematically for the special case of a *right regular* T-prism with the topology (connectivity) pictured. A right regular T-3 would have two equilateral triangular end faces, parallel to one another and normal to the axis joining their centers. Figure 2 depicts a sketched top view of such a T-3 prism, in which the upper face is ABC and the lower face is DEF. Cables connect the node pairs AE, BF, and CD, while bars connect the node pairs AD, BE, and CF. For that topology, the tensegrity geometry displays a relative twist angle between the end faces of 30 degrees. Figure 3 depicts the sketched top view of a right regular T-4 prism, for which the twist angle is 45 degrees.

BEHAVIOR OF TENSEGRITY PRISMS

The behavior of tensegrity structures is best introduced with a description of what is encountered physically when constructing such a T-3 prism. It is understood that

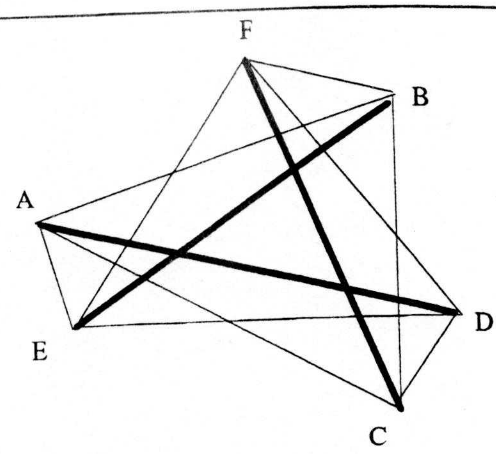

Figure 2. Top view of right regular T-3 prism

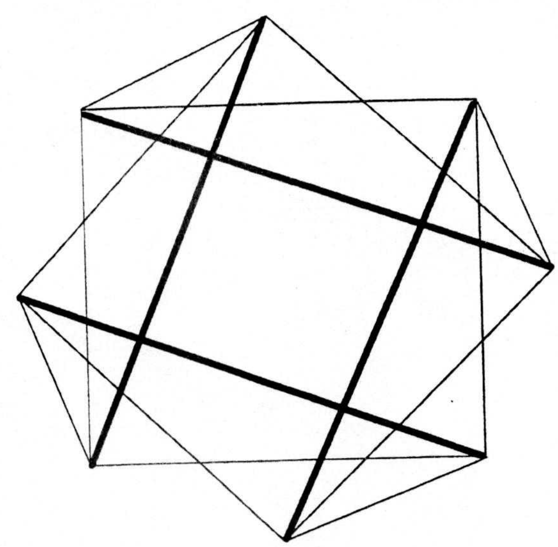

Figure 3. Top view of right regular T-4 prism

in the intended tensegrity geometry a prestress state will be encountered with all cables in tension and all bars in compression. For purposes of discussion it is useful to designate one cable as the last one to be constructed. All members except that last cable can be connected and the system will remain a loose bundle of bars and cables. The construction task then requires that the two nodes to be joined by the last cable be displaced towards one another in space, defining a line that will subsequently be occupied by that last cable.

The construction is first described for a structure in which the axial deformation of bars and cables is negligible. The displacement of the last pair of nodes is initially unrestrained, in that no structural resisting forces are developed at those nodes as the other cables and bars move with that displacement. However, it is found that at some specific separation length the two nodes can no longer approach one another, because the whole system has "snapped" to a geometry in which a three dimensional truss has been formed. Any attempt to bring the nodes closer together becomes a force (along that line of action) that acts to prestress all members in the truss. The structure has thereby reached its tensegrity geometry through a "form-finding" behavior. This is a dramatic self-erecting condition, and it is immediately observed that the tensegrity geometry achieved is unique. Every member length, except for that of the last cable, has been fixed, and it is found that there is only one unique length of the last cable at which the tensegrity structure will form, and there is only one unique geometry (set of nodal positions) at which that will occur. It is further noted that the tensegrity geometry is characterized by the physical prestress state, controlled by "tightening" that last cable. The resulting geometry is essentially stable, a point to be discussed more fully in a later section.

If the length of any one cable, or set of cables, is changed, then this physical exercize can be repeated and the last cable re-adjusted in length to reach a new tensegrity geometry. In so doing the nodes will take new positions in space. It is by such compatible changes in cable lengths that the T-prism can be made to operate like a VGT.

A very different physical sense of the problem is obtained if the T-prism is constructed with some subset of cables displaying large elastic elongations. It often proves easiest, and most revealing, to construct such prisms with springs ("rubber bands") for some or all of the cables. In this particular discussion, we might imagine that all cables in the end faces are inextensible, but that the longer cables AE, BF, and CD are constructed with such springs or rubber bands. In this construction, the form-finding occurs more gracefully. Most significantly, one can physically assert changes in the nodal positions, and the structure will automatically reshape itself into an accompanying tensegrity geometry. (When this behavior is experienced the tensegrity prism appears to be a highly adaptive structural form.) For example, if one purposely moves two nodes such as A and E closer together, this is equivalent to having replaced member (spring) AE with one of shorter initial length and/or greater stiffness, and the deformed length of members BF and CD will change accordingly and automatically as a form-finding behavior. In such cases, the structure moves through different tensegrity geometries as changes in nodal position are imposed.

These two different physical approaches to constructing a tensegrity prism reveal two alternative ways of posing their mathematics and their mechanics. In the first construction it is obvious that the unique tensegrity geometry (the set of nodal positions) represents a constraint satisfaction problem, in which the member lengths (other than that of the last cable) and the equilibrium conditions are the constraints. In that same construction it is also obvious that changes in nodal position require ongoing satisfaction of the same constraint problem as some member lengths are altered. In the second construction it is obvious that the tensegrity geometry corresponds to a state in which the stored energy in the elastic members is made stationary.

In discussing possible use for VGT applications, it is necessary to address the problem in its first construction. A known starting tensegrity geometry has been asserted, and in principle from that starting state every achievable T-3 geometry can be reached by suitably changing member lengths. However, we will show that identifying "suitable" (permissible) changes in member lengths remains a challenging problem in itself.

In summary, several distinctive characteristics of tensegrity structures motivate us to study their suitability for robotic applications, including the following:

- Tensegrity prisms can be totally self-erecting from a low-volume bundle by tensioning the last cable. Such a capability would be useful for constructing temporary structures. It could also be used for constructing a conventionally framed tower; a tensegrity tower could be self-erected, strong enough and stiff enough to support the weight of workers, who could then climb the tower and add additional structural members to create a conventional tower framing. Similarly, a self-erecting prism could be remotely maneuvered to some location within a piping system in its collapsed bundle state, and then self-erected to form an internal framework.

- Noting the large number of cables in any typical tensegrity structure, it is likely that they will frame a geometry with less total material weight than a conventional three-dimensional truss. The self-erecting property, together with the anticipated weight efficiency, suggests its applicability for on-orbit space robotics and space structures.

- When member lengths can be varied, or when large elastic elongations are admitted, tensegrity structures present a new family of machine kinematics and a new family of shape-adaptable structures, with the immense advantage that actuation is required only for tension members, requiring a far simpler technology than general bi-directional translational actuators.

PREVIOUS WORK

Tensegrity structures were demonstrated by Buckminster Fuller [1] and Snelson with a patent date of 1962. The famed literary critic Hugh Kenner [2] provided an insightful quantitative analysis of regular tensegrity prisms and spherical tensegrities as introductory chapters in his 1974 book on geodesic geometry. A related book by Pugh [3] is essentially an instruction manual for building hundreds of tensegrity structures. The fundamental contributions in mechanics are found in papers by Calladine [4-9] and his colleagues (Tarnai, Pellegrino, and others), and that work is the basis for most of the results reported here.

Mathematical studies of tensegrity frameworks have focused on characterization of rigid configurations, generalizing classical results on rigidity of pinned frameworks. The word tensegrity as used in the mathematics literature refers to structures which involve tension-only members (cables) and compression-only members (struts) as well as conventional structural bars. The Fuller-type tensegrity structures addressed here are only a subset of that class. Recent contributions are well characterized in the work of Roth and Whiteley [10], Connelly [11], and Connelly and Whiteley [12]. There is even a body of work in cell biology, described by Ingber [13], examining the mechanical behavior of the tensegrity prism as the basis of the cytoskeleton. However, our discussion in this paper will focus on analytical results for engineering applications to construction robotics.

ANALYSIS; EXAMPLE PROBLEMS

The analytical results are best described with reference to examples, and the T-3 and T-4 prisms will be used. We first examine these prisms by the conventional interpretation of Maxwell's rule. For a pin-connected framework in three-dimensional space, the determinacy measure is labelled k and is equal to (n-3j+6), where n is the number of members, j is the number of joints, and 6 reflects the number of independent rigid body motions in space; alternately, 6 is the minimum number of required reaction forces needed to create a conventional supported spatial structure. We make the following observations:

- The T-3 prism generically yields a determinacy measure k equal to zero, which conventionally describes a determinate truss. However, the T-3 prism is flexible (structurally unstable) in any placement other than the tensegrity position, in which it is stabilized by a prestress.
- The T-4 prism generically yields a determinacy measure k equal to -2, which conventionally describes an unstable structure, namely a machine with two independent (finite) mechanisms. However, the T-4 prism may assume a tensegrity position, in which it supports a stabilizing prestress.

The description of these observations is effected by consideration of the equilibrium (or statics) matrix [A]. The placement (nodal geometry) of a given structure determines the matrix [A] which relates nodal forces {p} to member forces {f} in the form {p}=[A]{f}, which is a generalized force-stress relationship. Dually, the nodal displacements {v} are infinitesimally related to the member elongations {e} by the matrix equation {e}=[B]{v}, which is a generalized strain-displacement relationship, where [B] is called the compatibility (or kinematics) matrix. From the contragradience principle of mechanics [B] is equal to the transpose of [A]. A *prestress* then is defined by the condition [A]{f}={0}. Similarly, an *infinitesimal flexure* is defined by the condition that the matrix product [B]{v} yield a vector of zeros, corresponding to nodal displacements without member elongation, or a vector with negative elongations for cable members, corresponding to nodal displacements with slackening of cable members. Equilibrium of a placement then is alternatively describable through the force-stress or the strain-displacement relation. For computations it is convenient to remove rigid-body motions from consideration. Most easily one introduces constraints on nodal displacements, For example, a structure essentially equivalent to the T-3 prism can be formed by making nodes D, E, and F into pinned supports, eliminating the cables DE, EF, and FD.

The chief distinguishing characteristic of T-prisms, as for any tensegrity structure, is that equilibrium requires prestressibility, and is accompanied by the presence of one or more infinitesimal kinematic mechanisms. An example with lower dimensionality is shown in Figure 4. In two dimensions as shown, a truss is constructed with

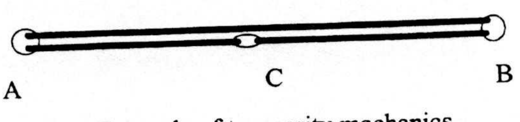

Figure 4. Example of tensegrity mechanics

three members and three joints. That topology would generically create a triangle, which would be a stable and determinate structure, one which would not sustain a prestress state and would not demonstrate any mechanism motion. However, in the specific geometry shown in

Figure 4 a prestress state is possible, exemplified by members AC and CB in tension with member AB in compression. Visualize support reactions vertically and horizontally at A and vertically at B, and then consider the two generalized nodal degrees-of-freedom at C in the absence of prestress. A force applied in the x-direction would be directly resolved by an axial force in member AC. However, a force applied in the y-direction would not be resolved until some displacement occurred; an infinitesimal kinematic mechanism is present. Specifically, the resulting structural stiffness (the force-displacement relationship for that nodal degree-of-freedom) in the y-direction is a non-linear geometric cubic stiffening with an initial slope of zero; that initial flat slope corresponds to the finding of an infinitesimal mechanism motion. However, the presence of prestress causes the initial slope of the force-displacement relationship to be positive. The behavior of the simple planar structure in Figure 4 exemplifies the behavior of the spatially complex tensegrity structures.

A proper, generalized interpretation of Maxwell's rule (see Calladine [4,8]) establishes that the determinacy measure denoted k is equal to (s-m), where s is the number of possible independent prestress modes and m is the number of possible independent kinematic mechanisms. For tensegrity structures these conditions supersede the more conventional definitions of necessary conditions for structural stability, determinacy, etc. For example, in a conventional structure with a determinacy measure k equal to zero, the number of possible prestress modes, s, and the number of kinematic mechanisms, m, are both equal to zero. In contrast, for the T-3 prism in its tensegrity geometry there is found to be one prestress mode and one infinitesimal mechanism, resulting in the same indeterminacy measure of zero. Similarly, the T-4 prism has one prestress mode and three (infinitesimal) mechanisms, totalling to the resulting indeterminacy measure of -2.

- The number of prestress modes and the number of infinitesimal mechanisms is found from the conventional application of Maxwell's rule *in combination with* the rank of [A]. Equilibrium (tensegrity) positions require existence of a prestress, which occurs in these T-prisms only when the [A] matrix is rank-deficient. For T-prisms the column dimension of [A] is always exceeded by the row-dimension (equal for T-3, strictly less for all others). In the examples considered the number of prestress modes s is equal to the rank deficiency and is observed to equal one. Therefore, for the T-3 prism the quantity m is equal to one, and for the T-4 prism the quantity m is equal to three. For higher order T-n prisms the quantity m is equal to (2n-6+1). The prestress modes and the mechanisms themselves are then expressed in the fundamental subspaces of [A]. Specifically, the {f} vector describing the member forces comprising the prestress mode forms the nullspace of [A], while the {v} vectors describing the nodal displacements comprising the kinematic mechanisms form the left nullspace of [A], most conveniently calculated as the nullspace of [B].

- Applied loads can be decomposed into those which are resisted within the equilibrium geometry and those which are resisted only after mechanism motion.

- For any general set of nodal positions, not a tensegrity geometry, a truss structure with T-3 topology would appear to be almost conventional in its behavior. By the ordinary application of Maxwell's rule it would be characterized as a determinate truss, and analysis by the method of joints or by use of the statics matrix would yield the member forces in terms of applied nodal forces. However, it is then recognized that a tensegrity geometry is a pathological state for such a truss. The determinant of the statics matrix becomes zero, a condition which must be present if rank deficiency occurs. More conventionally, the singularity of the statics matrix is interpreted to mean that applied nodal forces, in general, cannot be resolved by the truss members. In a related observation, an analysis attempt using the matrix stiffness method would be expected to fail because the stiffness matrix would be singular. Conventionally, this would all be interpreted to indicate that the structure is unstable. In a tensegrity geometry that "instability" is limited to the infinitesimal kinematic mechanisms.

- In the tensegrity prisms of interest, the mechanisms are infinitesimal rather than finite, such that they display a geometric stiffness (cubic force-displacement relationship) to resist applied loads. For the right regular T-3 prism, the mechanism consists of a rotation about the longitudinal axis of the upper triangular face with respect to the lower face, with a compatible axial displacement.

- Considering other tensegrity prisms of the same topological family, such as the T-4 depicted in Figure 3, it is instructive (and practical) to install them with support at each node in their "base." Upon doing so, eliminating the cables forming the edges to the base polygon and noting in passing that the support condition still exceeds the minimum set of required reaction components, the resulting tensegrity prism is found to have a square statics matrix [A] and to possess one prestress state and one kinematic mechanism, precise counterparts to the conditions found for the T-3.

Establishment of a prestress grants to each mechanism an initial constant stiffness (a linear force-displacement relationship) to resist applied loads.

In order for a tensegrity structure to carry loads, the prestress state should assure that all cables remain in tension as external loads are applied. Therefore, it is necessary to find the member forces under the applied loads and set the prestress level to overcome all compression forces induced in cables. If a nodal position is perturbed slightly the member forces can be found using conventional analysis tools.

- Most significantly for robotics one can utilize the {e}=[B]{v} relation to relate increments in member lengths to changes in nodal positions. This corresponds to the linearized inverse kinematics of the tensegrity prism as a manipulator. However, there are three major challenges. The first is that the matrix is not invertible, so that a direct solution for nodal motion given length changes is not possible. A second challenge is that finite changes in position cannot be calculated directly but must be obtained by a series of small steps. The most significant challenge, however, is that any change in position must be a movement into another tensegrity geometry, placing another constraint on the {e}=[B]{v} relation. The relationship asserted by the kinematics matrix [B] does not, in itself, assure a move to a new tensegrity geometry. We have begun construction of algorithms which will ensure that each incremental step carries into a new tensegrity geometry, ensuring that the prism may trace a path through neighboring tensegrity geometries when used as a manipulator.

- In principle, for a given topology, if a set of nodal positions is proposed it should be possible to check whether that set constitutes a tensegrity geometry by simply forming the statics matrix and seeing if it is rank deficient. However, in real terms the identification of a zero determinant is subject to the numerical conventions of the computational tool employed. In this study both MATLAB and Maple have been used for that purpose, but the analyst is advised to examine results carefully. Direct methods to guarantee determinant-zero conditions do not seem convenient, so we must have recourse to a secondary computation to recorrect incremental motions to attain the tensegrity position.

DISCUSSION OF EXPERIMENTAL RESULTS

These various results have all been verified by experimental study of a T-3 prism approximately 1m tall. It was constructed with cables demonstrating negligible material extensibility, but with manual turnbuckles in cables AE, BF, and CD to represent a controllable manipulator. The physical behavior conditions described herein were all observed, including manual control of the prism as a manipulator. The mathematical results derived from the statics matrix [A], as outlined in this paper, were all confirmed in comparison to the experimental observations. Structural analysis was undertaken with geometrically non-linear behavior modelled using the ABAQUS finite element analysis package, loaded by nodal forces in the direction of the kinematic mechanism displacements. With the structure in its "exact" tensegrity geometry, and in the absence of prestress, the analysis was found to fail as a consequence of the zero slope condition in the generalized force-displacement relationship. However, with any slight alteration in nodal position the analysis properly revealed the geometrically stiffening behavior as described earlier for the example in Figure 4. Similarly, in the presence of prestress the analysis properly revealed an initial stiffness as well as the subsequent effect of geometric stiffening. We note that a finite element analysis can also be made numerically stable by attaching very soft springs to the nodes to provide artificial support. The T-3 prism behavior was also studied using the Working Model 3-D application software, modelling the assembly step outlined earlier, in which the two nodes to be connected by the last cable are displaced toward one another. In some attempts, at some initial geometries, it successfully traced the motion of the bars and cables to the tensegrity geometry, but in other attempts it demonstrated numerical instability; use of that application software deserves further study.

SUMMARY

The physical behavior of tensegrity structures has been described, with particular attention to a family of prisms of minimal configuration referred to as T-prisms. The available analysis approaches have been discussed and example results have been outlined. An experimental study provided the opportunity for direct physical observations and for confirmation of analytical and numerical results.

ACKNOWLEDGEMENTS

The authors thank the Department of Architecture, the Department of Civil and Environmental Engineering, and the Department of Mathematics for the support of their research interests on this topic. The authors are particularly grateful to Mr. Yin Lau, a May 1997 graduate in Civil and Environmental Engineering, who performed the experimental studies as his Senior Project. E-mail contact is invited to ijo@ce.cmu.edu.

REFERENCES

1. Fuller, R.B., "Tensile-integrity structures," U.S.Pat. 3,063,521, 1962.

2. Kenner, H., *Geodesic Math and How to Use It*, University of California Press, 1976.

3. Pugh, A., *An Introduction to Tensegrity*, University of California Press, 1976.

4. Calladine, C.R., "Buckminster Fuller's 'Tensegrity' structures and Clerk Maxwell's rules for the construction of stiff frames, *International Journal of Solids and Structures*, Vol. 14, 1978.

5. Tarnai, T., "Simultaneous static and kinematic indeterminacy of space trusses with cyclic symmetry," *International Journal of Solids and Structures*, Vol. 16, 1980.

6. Tarnai, T., "Problems concerning spherical polyhedra and structural rigidity," *Structural Topology*, Vol. 4, 1980.

7. Pellegrino, S., and Calladine, C., "Two-step matrix analysis of prestressed cable nets," *Proceedings Third Internal Conference on Space Structures*, Guildford, 1984.

8. Pellegrino, S., and Calladine, C.R., "Matrix analysis of indeterminate frameworks," *International Journal of Solids and Structures*, Vol. 22, 1986.

9. Calladine, C.R., and Pellegrino, S., "First-order infinitesimal mechanisms," International Journal of Solids and Structures, Vol. 27, 1991.

10. Roth, B., and Whiteley, W., "Tensegrity frameworks," *Transactions American Mathematics Society*, Vol. 265, 1981.

11. Connelly, R., "Rigidity and energy," *Invent. Math.*, Vol. 66, 1982.

12. Connelly, R., and Whiteley, W., "Second-Order Rigidity and Prestress Stability for Tensegrity Frameworks," *SIAM J. Discrete Math.*, Vol. 9, 1996.

13. Ingber, D.E., "Cellular tensegrity: defining new rules of biological design that govern the cytoskeleton," *Journal of Cell Science*, Vol. 104, 1993.